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Titchmarsh–Weyl theory and Levinson's theorem for Dirac operators

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Abstract. For a Dirac system with a perturbation potential $V(r)$ obeying $\int_0^\infty (1+r)|V(r)|\,\mathrm{d}r < \infty$, the behaviour of the Titchmarsh–Weyl m-function near the spectral-gap endpoints is examined. As a consequence, we obtain another demonstration of the Levinson theorem.

1. Introduction

We consider the Dirac operators

$$H_0 = J[y' - B(r)y] \tag{1}$$

and

$$Hy = J[y' - (B(r) + P(r))y] \qquad 0 < r < \infty \tag{2}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B(r) = \begin{pmatrix} \mu/r & -E-M \\ E-M & -\mu/r \end{pmatrix}$$

$$P(r) = \begin{pmatrix} 0 & V \\ -V & 0 \end{pmatrix} \qquad \mu = \pm 1, \pm 2, \pm 3, \ldots \qquad M > 0$$

and

$$y(E,r) = \begin{pmatrix} y_1(E,r) \\ y_2(E,r) \end{pmatrix}.$$

Our main assumption will be the limit-point hypothesis

$$\int_0^\infty (1+r)|V(r)|\,\mathrm{d}r < \infty. \tag{3}$$

Operators of the form (1) and (2), with various hypotheses, have recently attracted considerable attention [1–7] in connection with Levinson's theorem [8], by which the number of bound states of H are obtained by variation of an appropriate phase along the continuous portion of the spectrum. It is a purpose of this paper to provide another demonstration of this celebrated theorem. The contribution here is the method of proof. In [5] and [6], Green's function methods are used to obtain the Levinson theorem, in [1–3], Jost functions are employed and, in [4], the Sturm–Liouville theorem is used. Our present treatment will rely on the Titchmarsh–Weyl m-function (see [9–11] for the regular case) associated with the equation

$$Hy = Ey \tag{4}$$

which we view as a perturbation of

$$H_\circ y = Ey. \tag{5}$$

The spectrum of H is known to consist of an absolutely continuous part $(-\infty, -M] \bigcup [M, \infty)$ with a possible finite number of eigenvalues in $(-M, M)$; and, hence, Levinson's theorem consists of varying an appropriate argument around $\pm M$; from here also stems our interest in studying the m-function as $E \to \pm M$, whose behaviour also characterizes the existence of bound states (BS) or half bound states (HBS). Our results are only stated for the case $E \to +M$ as the extension to $E \to -M$ is obvious.

Our analysis will draw a lot on [2, 3]. In particular, our asymptotics are obtained in the same manner as in these references. As such, we will give a number of asymptotic results without detail and refer the reader accordingly. The paper is organized as follows. In section 2, we obtain the Titchmarsh–Weyl m-function in terms of Jost-like functions. We then derive the m-function asymptotics and obtain the Levinson theorem in section 3.

2. Jost functions and m-functions

Here we shall describe the m-functions and obtain their representations in terms of Jost functions. Let us begin with the fundamental matrix for (5), i.e.

$$Y_\mu^0(E,r) = kr \begin{pmatrix} [n_\mu(k) j_{\mu-1}(kr) - j_\mu(k) n_{\mu-1}(kr)] & \dfrac{E+M}{k}[j_{\mu-1}(k) n_{\mu-1}(kr) - n_{\mu-1}(k) j_{\mu-1}(kr)] \\ \dfrac{k}{E+M}[n_\mu(k) j_\mu(k) n_\mu(kr)] & [j_{\mu-1}(k) n_\mu(kr) - n_{\mu-1}(k) j_\mu(kr)] \end{pmatrix} \tag{6}$$

where $n_\mu(x)$ and $j_\mu(x)$ are the spherical Bessel functions and $k = \sqrt{E^2 - M^2}$ is chosen so that $k > 0$ on $M < E < \infty$ and $k < 0$ on $-\infty < E < -M$. In particular, $Y_\mu^0(E, 1) = I_2$. Then we partition $Y_\mu^0(E,r)$ as $[\theta_\mu^0(E,r)\ \phi_\mu^0(E,r)]$. The m-functions for (5) at $r = 0$ and $r = \infty$ are then, respectively, given by

$$m_-^0(E) = -\lim_{r\to 0} \frac{\theta_1(E,r)}{\phi_1(E,r)} \tag{7}$$

and

$$m_+^0(E) = -\lim_{r\to\infty} \frac{\theta_1(E,r)}{\phi_1(E,r)}. \tag{8}$$

Straightforward computation using (6) yields, as $r \to \infty$ on $\operatorname{Im} k > 0$,

$$\exp(\mathrm{i}(kr - \pi\mu/2))Y^0_\mu(E,r) = \frac{k}{2}\begin{pmatrix} h_\mu(k) & -\frac{E+M}{k}h_{\mu-1}(k) \\ \frac{\mathrm{i}k}{E+M}h_\mu(k) & -\mathrm{i}h_{\mu-1}(k)\end{pmatrix} \tag{9}$$

where $h_\mu(x) = n_\mu(x) + \mathrm{i}j_\mu(x)$; and as $r \to 0$, we have

$$(kr)^\mu Y^0_\mu(E,r)$$
$$= k\begin{pmatrix} \frac{(kr)^{2\mu}n_\mu(k)}{\Gamma_{-3}} + kr\Gamma_3 j_\mu(k) & -\frac{E+M}{k}\left[kr\Gamma_3 j_{\mu-1}(k) + \frac{(kr)^{2\mu}n_{\mu-1}(k)}{\Gamma_1}\right] \\ \frac{k}{E+M}\left[\frac{(kr)^{2\mu+1}}{\Gamma_{-1}} + j_\mu(k)\Gamma_1\right] & -\Gamma_1 j_{\mu-1}(k) - \frac{(kr)^{2\mu+1}n_{\mu-1}(k)}{\Gamma_{-1}}\end{pmatrix}$$
$$+ \mathrm{o}(1) \tag{10}$$

where $\Gamma_a \equiv (2\mu - \alpha)!! = (2\mu-\alpha)(2\mu-\alpha-2)(2\mu-\alpha-4)\cdots$. From (7) and (9), the m-functions for (5) are therefore given by

$$m^0_-(E) = \frac{k}{E+M}\frac{j_\mu(k)}{j_{\mu-1}(k)} \tag{11}$$

and

$$m^0_+(E) = \frac{k}{E+M}\frac{h_\mu(k)}{h_{\mu-1}(k)}. \tag{12}$$

Next, we look at solutions and m-functions for (4). By variation of parameters, the solution $y(E,r)$ obeying

$$y(E,1) = \begin{pmatrix} y_1(1) \\ y_2(1)\end{pmatrix}$$

is given by

$$y(E,r) = Y^0_\mu(E,r)\left[\begin{pmatrix} y_1(1) \\ y_2(1)\end{pmatrix} + \int_1^r [Y^0_\mu(E,t)]^{-1}P(t)y(t)(E,t)\,\mathrm{d}t\right]. \tag{13}$$

Now let $[Y^0_\mu(E,t)]^{-1}P(t)y(E,t)$ be denoted by $Q(y(t))$ and let

$$A_y(E) = \frac{k}{2}\left(h_\mu(k), -\frac{E+M}{k}h_{\mu-1}(k)\right)\left[\begin{pmatrix} y_1(1) \\ y_2(1)\end{pmatrix} + \int_1^\infty Q(y(t))\,\mathrm{d}t\right]. \tag{14}$$

Then (9) and standard asymptotics yield that, as $r \to \infty$ for $\operatorname{Im} k > 0$, we have

$$\exp(\mathrm{i}(kr - \pi\mu/2))y(E,r) = A_y(E)\begin{pmatrix} 1 \\ \frac{\mathrm{i}k}{E+M}\end{pmatrix} + \mathrm{o}(1). \tag{15}$$

In particular, (15) holds for the solutions $\theta(E,r)$ and $\phi(E,r)$ defined by $[\theta(1,r), \phi(1,r)] = I$ and hence, the m-function for (4) at $r = \infty$ is obtained as

$$m_+(E) = -\lim_{r\to\infty}\frac{\theta_1(E,r)}{\phi_1(E,r)} = \frac{A_\theta(E)}{A_\phi(E)}. \tag{16}$$

Letting $B_y(E)$ be defined as

$$B_y(E) = k\Gamma_3 \left(j_\mu(k), -\frac{E+M}{k} j_{\mu-1}(k) \right) \left[\begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix} - \int_0^1 Q(y(t)) \right] \qquad (17)$$

we find that, as $r \to 0$ for $\operatorname{Im} k > 0$, we have

$$(kr)^\mu y(E, r) = B_y(E) \begin{pmatrix} r \\ \frac{\Gamma_3}{\Gamma_1} \frac{E+M}{k} \end{pmatrix} + o(r). \qquad (18)$$

It therefore follows from (18) that the m-function for (4) at $r = 0$ is

$$m_-(E) = -\lim_{r \to 0} \frac{\theta_1(E, r)}{\phi_1(E, r)} = -\frac{B_\theta(E)}{B_\phi(E)}. \qquad (19)$$

A few comments regarding $A_y(E)$ and $B_y(E)$, which we call Jost functions, are in order. It follows from (15) and (18) that (A_θ, A_ϕ) and (B_θ, B_ϕ) are non-vanishing pairs; for if either pair vanishes, then (4) has all L^2 solutions at the appropriate endpoint. In particular, if we think of the operator H as the union $T_0 \bigcup T_\infty$, where T_0 and T_∞ denote the restrictions of H to (0,1] and $[1, \infty)$, respectively, then the discete spectra of T_0 and T_∞ are the zeros of B_ϕ and A_ϕ, respectively.

Let us recall [9] that the m-function for (4) on $(0, \infty)$ is defined as

$$m(E) = \frac{1}{m^- - m^+} \begin{pmatrix} 1 & \frac{1}{2}(m^- + m^+) \\ \frac{1}{2}(m^- + m^+) & m^- m^+ \end{pmatrix}. \qquad (20)$$

Also, if we let $\rho(E)$ denote the spectral matrix function for H, then we recall that m and ρ are connected by the Titchmarsh–Kodaira formula

$$\rho(\lambda_2) - \rho(\lambda_1) = -\frac{1}{\pi} \lim_{\nu \downarrow 0} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} m(\mu + i\nu)\, d\mu \qquad (21)$$

at points of continuity λ_1, λ_2 of ρ. We shall use this connection to obtain the behaviour of ρ as $E \to M$. On account of (16) and (19), we may write (20) as

$$m(E) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \qquad (22)$$

where

$$m_{11} = \frac{A_\phi(E) B_\phi(E)}{F(E)} \qquad m_{22} = \frac{A_\theta(E) B_\theta(E)}{F(E)}$$

$$m_{12} = m_{21} = \frac{A_\theta(E) B_\phi(E) + A_\phi(E) B_\theta(E)}{2F(E)}$$

$$F(E) = A_\theta(E) B_\phi(E) - A_\phi(E) B_\theta(E). \qquad (23)$$

There are two other solutions ψ and f, relevant to our discussion, which we will call the Jost solutions. These are defined by their behaviours as $r \to 0$ and $r \to \infty$, respectively, namely

$$\psi(E, r) \to r^\mu \begin{pmatrix} \frac{1}{\Gamma_1} \\ \frac{(E-M)r}{\Gamma_{-1}} \end{pmatrix} \qquad (24)$$

and

$$f(E,r) \to \exp(\mathrm{i}(kr+\pi\mu/2))k^{\mu}\begin{pmatrix}\frac{\mathrm{i}k}{E-M}\\ 1\end{pmatrix} \tag{25}$$

and are explicitly given by

$$\psi(E,r) = r^{\mu}\begin{pmatrix}\frac{1}{\Gamma_1}\\ \frac{(E-M)r}{\Gamma_{-1}}\end{pmatrix} + Y^0_{\mu}(E,r)\int_0^r Q(\psi(t))\,\mathrm{d}t \tag{26}$$

and

$$f(E,r) = k^{\mu}\begin{pmatrix}\frac{E+M}{k}krh_{\mu-1}(kr)\\ krh_{\mu}(kr)\end{pmatrix} - Y^0_{\mu}(E,r)\int_r^{\infty} Q(f(t))\,\mathrm{d}t. \tag{27}$$

By evaluating Wronskian determinants, straightforward calculations then yield

$$A_y(E) = W[f(E,r);\, y(E,r)] \tag{28}$$

and

$$B_y(E) = W[\psi(E,r);\, y(E,r)] \tag{29}$$

where $y(E,r)$ is given by (13). We therefore obtain the relations

$$f(E,r) = A_{\phi}(E)\theta(E,r) + A_{\theta}(E)\phi(E,r) \tag{30}$$

and

$$\psi(E,r) = B_{\phi}(E)\theta(E,r) + B_{\theta}(E)\phi(E,r). \tag{31}$$

In particular, we have that

$$F(E) = W[\psi(E,r);\, f(E,r)] \tag{32}$$

where $F(E)$ is given by (23).

3. Asymptotics and Levinson's theorem

To obtain the m-function asymptotics, we shall need the solutions of equation (4) at $E = M$, i.e.

$$Hy = My \tag{33}$$

corresponding to those given in the last section. These are given by (13), (26) and (27) at $E = M$:

$$y(M,r) = Y^0_{\mu}(M,r)\left[\begin{pmatrix}y_1(M,1)\\ y_2(N,1)\end{pmatrix} + \int_1^r [Y^0_{\mu}(M,t)]^{-1}P(t)y(M,t)\,\mathrm{d}t\right] \tag{34}$$

$$\psi(M,r) = \begin{pmatrix}\frac{r^{\mu}}{\Gamma_1}\\ 0\end{pmatrix} + \int_0^r Q^0(\psi(t))\,\mathrm{d}t \tag{35}$$

and

$$f(M,r) = \exp(\mathrm{i}(kr+\mu\pi/2))k^{\mu}\begin{pmatrix}\frac{\mathrm{i}k}{E-M}\\ 1\end{pmatrix} + \int_r^{\infty} Q^0(f(t))\,\mathrm{d}t \tag{36}$$

where we have denoted $Q^0(z(t)) = [Y_{\mu}^0(M,t)]^{-1}P(t)z(M,t)$ and note that

$$Y_{\mu}^0(M,t) = \begin{pmatrix} r^{\mu} & \frac{2M\Gamma_3}{\Gamma_1}(r^{-\mu+1}-r^{\mu}) \\ 0 & r^{-\mu}\end{pmatrix}. \tag{37}$$

The corresponding Jost functions are given by

$$A_y(M) = (1, -2M\Gamma_3/\Gamma_1)\left(\begin{pmatrix} y_1(M,1)\\ y_2(M,1)\end{pmatrix} + \int_1^{\infty} Q^0(y(t))\,\mathrm{d}t\right) \tag{38}$$

and

$$B_y(M) = (0,1)\left(\begin{pmatrix} y_1(M,1)\\ y_2(M,1)\end{pmatrix} - \int_0^{1} Q^0(y(t))\,\mathrm{d}t\right) \tag{39}$$

which also have the identifications (28) and (29) and verify relations (30) and (31) *mutatis mutandis*. The following lemma is then obtained in the same way as lemma 2.1 of [4].

Lemma 3.1. Suppose that $V(r)$ obeys (3) and let $\delta > 0$. Then, there is a constant C depending only on δ such that, for $E \in [M, M+\delta]$:

(a) if $A_y(M) = 0$ and $\mu \geqslant \frac{3}{2}$, then

$$|y_1(E,r) - y_1(M,r)| \leqslant Ck^2\left(\frac{r}{1+kr}\right)^{\mu}$$

and

$$|y_2(E,r) - y_2(M,r)| \leqslant Ck^2\left[\left(\frac{r}{1+kr}\right)^{\mu+1} + \left(\frac{r}{1+kr}\right)^{\mu}\right]$$

(b) if $A_y(M) = 0$ and $\mu < \frac{3}{2}$, then

$$|y_i(E,r) - y_i(M,r)| \leqslant Ck^2\left[\left(\frac{kr}{1+kr}\right)^{2} + \frac{k^2 r}{1+kr}\right] \qquad i = 1,2.$$

Using lemma 3.1, we are then able to prove the next result, the proof of which follows that of theorem 2.2 of [3] (cf lemma 3.2b of [2]).

Lemma 3.2. The asymptotic behaviour of $A_y(E)$ and $B_y(E)$, uniformly on $0 \leqslant \arg(E-M) \leqslant 2\pi$, is as follows.

(a)

(i) If $A_y(M) = 0$ and $\mu \geqslant \frac{3}{2}$, then $A_y(E) = a_y^{>}k^2 + \mathrm{o}(k^2)$.

(ii) If $A_y(M) = 0$ and $\mu < \frac{3}{2}$, then $A_y(E) = a_y^{<}k + \mathrm{o}(k)$, where

$$a_y^{>} = \frac{\Gamma_1}{2Md_y}\|y(M,\cdot)\|^2 \qquad a_y^{<} = -2\mathrm{i}Md_y \qquad d_y = -\int_1^{\infty} t^{\mu}V(t)y_1(M,t)\,\mathrm{d}t.$$

(b) If $B_y(M) = 0$, then $B_y(E) = b_y k^2 + o(k^2)$, where

$$b_y = \int_1^\infty t^\mu V(t) \left[y_1(M,t) - \frac{2M}{\Gamma_{-1}} t y_2(M,t) \right] \mathrm{d}t.$$

Recalling (32), we may evaluate Wronskians using (26), (27), (35) and (36) to obtain the following representations:

$$F(E) = \left(1, k^{2\mu} \frac{E+M}{\mathrm{i}k}\right) \left(\psi(E,0) + \int_0^\infty Q(\psi(t))\,\mathrm{d}t\right) \tag{40}$$

and

$$F(M) = 1 + \int_0^\infty V(t) t [\Gamma_1 \psi_2(M,t) + 2M\Gamma_3 t \psi_1(M,t)]\,\mathrm{d}t. \tag{41}$$

Standard asymptotics then yield the following lemma (we refer the reader to [2] for its proof).

Lemma 3.3. The asymptotic behaviour of $F(E)$ as $E \to M$, uniformly for $0 \leqslant \arg(E-M) \leqslant 2\pi$, is as follows.

(a) If $F(M) = 0$ and $\mu \geqslant \frac{3}{2}$, then $F(E) = f^{>} k^2 + o(k^2)$.

(b) If $F(M) = 0$ and $\mu < \frac{3}{2}$, then $F(E) = f^{<} k^2 + o(k^2)$, where

$$f^{>} = \frac{\Gamma_1}{2Mg} ||\psi(M,\cdot)||^2 \qquad f^{<} = -2M\mathrm{i}g \qquad g = -\int_0^\infty V(t) t^\mu \psi_1(M,t)\,\mathrm{d}t.$$

(c) In fact, one has $f^{<>} = (A_\theta(M) b_\phi + B_\phi(M) a^{<>}) - (A_\phi(M) b_\theta + B_\theta(M) a_\phi^{<>})$.

We now prove the following theorem.

Theorem 3.4. The asymptotic behaviour of $m(E)$ as $E \to M$ is as follows.

(a) $F(M) = 0 \iff \lim_{\nu \downarrow 0} \nu m(M + \mathrm{i}\nu) = S^{>}$, $\mu \geqslant 2$.

(b) $F(M) = 0 \iff \lim_{\nu \downarrow 0} \nu^{1/2} m(M + \mathrm{i}\nu) = S^{<}$, $\mu < 2$, where $S^{<}$ and $S^{>}$ are matrix functions of M and μ only.

Proof. This follows directly from lemmas 3.2 and 3.3 and the m-function representation (21). First, we note that the numerators in (22), evaluated at $E = M$, cannot simultaneously vanish. Hence, if $F(M) \neq 0$, we then have $m(M + \mathrm{i}\nu) = S_\circ(M)/F(M)$, where $S_\circ(\cdot)$ is a non-zero matrix, so that $\nu^{1/2} m(M + \mathrm{i}\nu) \to 0$ and $\nu m(M + \mathrm{i}\nu) \to 0$. If $F(M) = 0$ and $\mu \geqslant \frac{3}{2}$, we then obtain $m(M + \mathrm{i}\nu) = S_\circ(M)/f^{>} k^2$ and, noting that $k = 2\mathrm{i}M\nu^{1/2} + O(\nu)$ as $\nu \to 0$, we therefore obtain (a). For $F(M) = 0$ and $\mu < \frac{3}{2}$, we similarly have $m(M + \mathrm{i}\nu) = S_\circ(M)/f^{<} k$ and, hence, (b). □

Remarks.

(i) Theorem 3.4(a) precisely states that $E = M$ is a bound state (recalling that a half-bound state does not occur at $E = M$ for $\mu \geqslant 2$) provided $m(M + \mathrm{i}\nu)$ approaches $S^{>}/\nu$ as $\nu \downarrow 0$, which says m becomes singular at $E = M$ with a simple pole. This is expected since the poles of m, all of which are simple, are precisely the bound states of H.

(ii) Part (b) of theorem 3.4 says that $E = M$ is a half-bound state provided $m(M + \mathrm{i}\nu)$ approaches $S_\circ/\nu^{1/2}$ as $\nu \downarrow 0$, which is the same as the behaviour obtained for the Dirac system with non-singular potentials [9–11].

As an immediate corollary of theorem 3.4 and the well known identity (21), we therefore obtain the following corollary.

Corollary 3.5.

$$\text{(a)}\quad \lim_{E\downarrow M}\frac{\mathrm{d}\rho(E)}{\mathrm{d}E}=0 \text{ if no BS or HBS occur at } E=M$$

$$\text{(b)}\quad \lim_{E\downarrow M}\frac{\mathrm{d}\rho(E)}{\mathrm{d}E}=\frac{S_\circ(M)}{4\pi M^2 f^{>}(E-M)} \text{ if a BS occurs at } E=M$$

$$\text{(c)}\quad \lim_{E\downarrow M}\frac{\mathrm{d}\rho(E)}{\mathrm{d}E}=\frac{S_\circ(M)}{2\pi f^{<}M(E-M)^{1/2}} \text{ if a HBS occurs at } E=M.$$

By (32) and the fact that ψ and f are the only square integrable solutions at $r=0$ and $r=\infty$, respectively, we see that the zeros of $F(E)$ are precisely the bound states of H, excepting $E=\pm M$. If we let $\delta(E)$ denote the phase angle of $F(E)$, i.e. let $F(E)=|F(E)|\exp(\mathrm{i}\delta(E))$, we then have the following theorem, whose proof follows from the standard argument principle (exactly as in [2], noting that $M\to -M$) and the obvious extension of lemma 3.3 to $E=-M$.

Theorem 3.6 (Reference [2] theorem (1.1)). Let N_μ denote the number of bound states of H in $[-M,M]$. Then:

(a) $N_\mu=\frac{1}{\pi}(\delta_\mu(M)+\delta_\mu(-M))$, $\mu\geqslant\frac{3}{2}$; and

(b) $N_\mu=\frac{1}{\pi}(\delta_\mu(M)+\delta_\mu(-M))+\frac{\Delta}{2\pi}$, $\mu<\frac{3}{2}$, where

$$\Delta=\begin{cases}0 & \text{if a HBS does not occur at } E=M\\ -\pi & \text{if a HBS occurs at } E=M.\end{cases}$$

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